



Dynamic Mode Decomposition for Compressive System Identification

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Dynamic mode decomposition has emerged as a leading technique to identify spatiotemporal coherent structures from high-dimensional data, benefiting from a strong connection to nonlinear dynamical systems via the Koopman operator. In this work, two recent innovations that extend dynamic mode decomposition to systems with actuation and systems with heavily subsampled measurements are integrated and unified. When combined, these methods yield a novel framework for compressive system identification. It is possible to identify a low-order model from limited input–output data and reconstruct the associated full-state dynamic modes with compressed sensing, adding interpretability to the state of the reduced-order model. Moreover, when full-state data are available, it is possible to dramatically accelerate downstream computations by first compressing the data. This unified framework is demonstrated on two model systems, investigating the effects of sensor noise, different types of measurements (e.g., point sensors, Gaussian random projections, etc.), compression ratios, and different choices of actuation (e.g., localized, broadband, etc.). In the first example, this architecture is explored on a test system with known low-rank dynamics and an artificially inflated state dimension. The second example consists of a real-world engineering application given by the fluid flow past a pitching airfoil at low Reynolds number. This example provides a challenging and realistic test case for the proposed method, and results demonstrate that the dominant coherent structures are well characterized despite actuation and heavily subsampled data.

Nomenclature

A, \tilde{A}	= state transition matrix for x and \tilde{x}
A_Y, \tilde{A}_Y	= state transition matrix for y
B, \tilde{B}, \hat{B}	= actuation matrix
\hat{B}	= estimate of B
b	= vector of dynamic mode decomposition amplitudes
C	= output measurement matrix
F	= discrete cosine transform
G	= augmented matrix of A and B
K	= number of nonzero discrete cosine transform coefficients
m	= number of data snapshots
n	= dimension of the state
P	= projection matrix acting on x
p	= dimension of the measurement or output variable
q	= spatial coordinate
r, \tilde{r}	= rank of truncated singular-value decomposition of X, Ω
s	= discrete cosine transform coefficients of x
t	= time

t_k	= k th discrete time step
U, \hat{U}, \tilde{U}	= left singular vectors (proper orthogonal decomposition modes) of X, X', Ω
$U_Y, \hat{U}_Y, \tilde{U}_Y$	= left singular vectors (proper orthogonal decomposition modes) of Y, Y', Ω_Y
u	= control input, $u \in \mathbb{R}^q$
V, \hat{V}, \tilde{V}	= right singular vectors of X, X', Ω
$V_Y, \hat{V}_Y, \tilde{V}_Y$	= right singular vectors of Y, Y', Ω_Y
W, \tilde{W}_Y	= eigenvectors of \tilde{A}, \tilde{A}_Y
X	= data matrix of states, $X \in \mathbb{R}^{n \times m}$
X'	= shifted data matrix, $X' \in \mathbb{R}^{n \times m}$
\tilde{X}	= data matrix of reduced states, $\tilde{X} \in \mathbb{R}^{r \times m}$
x	= state vector, $x \in \mathbb{R}^n$
\tilde{x}	= low-rank state vector, $\tilde{x} \in \mathbb{R}^r$
Y	= data matrix of measurements, $Y \in \mathbb{R}^{p \times m}$
Y'	= shifted matrix of measurements, $Y' \in \mathbb{R}^{p \times m}$
y	= output vector, $y \in \mathbb{R}^p$
Δt	= time step
η	= noise magnitude
$\Theta = C\Psi$	= product of measurement matrix and sparsifying basis
Λ, Λ_Y	= matrix of dynamic mode decomposition eigenvalues of A, A_Y
λ	= dynamic mode decomposition eigenvalue of A
ν	= wave number
ξ	= spatial variable
$\Sigma, \hat{\Sigma}, \tilde{\Sigma}$	= matrix of singular values of X, X', Ω
$\Sigma_Y, \hat{\Sigma}_Y, \tilde{\Sigma}_Y$	= matrix of singular values of Y, Y', Ω_Y
Φ, Φ_Y	= matrix of dynamic mode decomposition with control modes of A, A_Y
Φ_S	= matrix of sparse representation of Φ
$\hat{\Phi}$	= compressed sensing estimate of Φ
Ψ	= orthonormal basis (e.g., Fourier or proper orthogonal decomposition)
Ω	= state and input snapshot matrices $[X^T \ Y^T]^T$

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Ω_Y	=	output and input snapshot matrices $[Y^T \quad Y^T]^T$
ω	=	continuous-time dynamic mode decomposition eigenvalue of A
Y	=	data matrix of control inputs

I. Introduction

DYNAMIC mode decomposition (DMD) is a dimensionality reduction technique introduced by Schmid [1] in the fluid dynamics community to decompose high-dimensional fluid data into dominant spatiotemporal coherent structures [1–7]. Shortly after its introduction, DMD was reframed by Rowley et al. [2] as a numerical technique to approximate the Koopman operator [8–11], establishing a strong connection to the analysis of nonlinear dynamical systems. Unlike other dimensionality reduction techniques, such as proper orthogonal decomposition (POD) [12], DMD is designed to extract modes that are spatially coherent, oscillate at a fixed frequency, and grow or decay at a fixed rate. Thus, DMD yields a set of modes along with a linear evolution model. Since being introduced in fluid dynamics, DMD has been widely applied in fields as diverse as epidemiology [13], neuroscience [14], robotics [15], video processing [16], and financial trading [17].

DMD may be thought of as a form of system identification, resulting in reduced-order models that are more tractable than the original high-dimensional dynamics. Low-order models are especially important for the control of high-dimensional dynamical systems, such as fluid flow control [18,19], where fast control decisions must be enacted to reduce latency for robust performance. There are a number of excellent overviews of model reduction [20] and system identification [21,22] for such high-dimensional systems. Balanced truncation provides a principled approach to reducing the system dimension by identifying a reduced-order model with the most jointly controllable and observable states [23], and extensions include the balanced proper orthogonal decomposition (BPOD) for systems with very large state dimension [24,25]. It was recently shown [26] that BPOD is equivalent to the eigensystem realization algorithm (ERA) [27]. In Tu et al. [4], it was further shown that DMD may be equivalent to ERA under certain conditions. Proctor et al. [28] further extended DMD to include inputs and control, disambiguating internal state dynamics from the effect of actuation. They also established deeper connections to the related subspace identification methods [29–33]. DMD and DMD with control may be thought of as being in the family of subspace identification techniques, but particularly with a perspective toward high-dimensional fluid dynamical systems and with theoretical connections to Koopman analysis. Since its introduction, DMD has been extensively studied and rigorously benchmarked and compared with several other leading techniques in dimensionality reduction and reduced-order modeling; see, for example, [5,28,34,35].

DMD relies on the fact that even high-dimensional dynamics typically evolve on a low-dimensional attractor. This low-dimensional behavior suggests sparsity in an appropriate basis, so that sparsity-promoting and randomized techniques may be exploited to reduce measurement resolution, bandwidth requirements, and computational overhead. Sparsity was first used in DMD by Jovanović et al. [36] to select the dominant DMD modes. Compressed sensing [37–39] was subsequently used to compute DMD using snapshots that were sampled below the Shannon–Nyquist sampling limit in time [40] and in space [41,42]. It was shown in Brunton et al. [41] that it is possible to reconstruct accurate DMD modes with surprisingly few spatial measurements, and if full-state data are available, performing DMD on compressed data dramatically reduces computation time. In addition to compressed sensing, randomized linear algebra has been leveraged to accelerate DMD computations [43,44]. Recently, Kalman-filter-based DMD has also been proposed in [45,46], which demonstrated improved performance in tracking eigenmodes in the presence of noise.

In this work, we combine the compressed sensing DMD [41] and DMD with control [28] approaches, resulting in a powerful mathematical framework for compressive system identification. There has been other work on compressive system identification [47–55], but this

is the first effort in the context of DMD; in a related work, sparse representation in a library of DMD modes was used for regime classification [56]. We begin by presenting the DMD algorithm and related compressed sensing and control extensions in a common framework and notation, making it possible to unify these approaches. Combining these algorithms results in the compressive DMD with control (cDMDc) architecture. cDMDc then makes it possible to identify reduced-order models on downsampled spatial measurements of a high-dimensional system and then reconstruct the full-state dynamic modes associated with the model using compressed sensing. This approach adds interpretability to otherwise black-box system identification models because the reduced states may now be associated with full-state modes. DMD and variants have been extensively explored and compared against other leading methods in past work. Thus, in this work, we focus on demonstrating that the cDMDc algorithm faithfully reproduces the results from the full-state DMDc algorithm, which will serve as the baseline. In addition, theoretical results are demonstrated for linear systems, where the DMD framework has rigorous guarantees. The resulting cDMDc architecture is demonstrated on two example dynamical systems, including a high-dimensional fluid flow simulation.

II. Background: The Dynamic Mode Decomposition

Matrix decomposition techniques are ubiquitous in the data sciences. Their fundamental objective is often to extract low-rank and interpretable patterns from data. Foremost among matrix decomposition methods is singular-value decomposition (SVD), which is the computational engine for principal component analysis and produces a set of ranked orthonormal modes that capture the dominant correlation structures in the data. However, the SVD fails to correlate both spatial and temporal features of the data together. The DMD provides a least-square regression architecture whereby both space and time are jointly correlated by merging a spatial SVD with a temporal Fourier transform [3]. Specifically, the DMD algorithm decomposes the data matrix X into the rank r approximation

$$X \approx \Phi \Lambda b \quad (1)$$

where the columns of Φ are the DMD modes (spatial structures), Λ is a Vandermonde matrix composed from the DMD eigenvalues λ_j , and the vector b determines the weighting of each of the r modes. Thus, each spatial mode in Φ is associated with a single temporal eigenvalue λ .

The DMD method was originally used as a low-rank diagnostic tool for decomposing fluid flow data into dominant spatiotemporal modes [1,2,4,5], providing a valuable interpretation of coherent structures in complex systems. Recent innovations have also made significant progress to employ DMD for robust future-state prediction [36,58] and control for input–output systems [28], even when using only a small number of measurements [41].

The exact DMD algorithm formulates the decomposition as a least-squares regression. Specifically, a series of snapshots $\mathbf{x}_k \in \mathbb{R}^n$ sampled at discrete instances in time $t_k, k = 0, \dots, m$ are arranged into the data matrix

$$X = \begin{bmatrix} | & | & \dots & | \\ \mathbf{x}_0 & \mathbf{x}_1 & \dots & \mathbf{x}_{m-1} \\ | & | & \dots & | \end{bmatrix} \quad (2)$$

and the time-shifted matrix

$$X' = \begin{bmatrix} | & | & \dots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_m \\ | & | & \dots & | \end{bmatrix} \quad (3)$$

Typically $n \gg m$ (i.e., there are many more spatial measurements available than temporal). The vector \mathbf{x}_k is the state of a high-

Algorithm 1 Exact DMD [4]

Input: Data matrix X , shifted data matrix X' , and target rank r .
Output: DMD spectrum Λ and modes Φ .
1: **procedure** DMD(X, X', r)
2: $[U, \Sigma, V] \leftarrow \text{SVD}(X, r)$ ▷ Truncated r -rank SVD of X .
3: $\tilde{A} \leftarrow U^* X' V \Sigma^{-1}$ ▷ Low-rank approximation of \tilde{A} .
4: $[W, \Lambda] \leftarrow \text{EIG}(\tilde{A})$ ▷ Eigendecomposition of \tilde{A} .
5: $\Phi \leftarrow X' V \Sigma^{-1} W$ ▷ DMD modes of \tilde{A} .
6: **end procedure**
Note that if $\lambda_i = 0$, then $\phi_i = U w_i$ for step 5. In the original DMD algorithm [57] all modes are computed as $\phi_i = U w_i$.

dimensional system such as a fluid flow. Here, we assume evenly sampled snapshots, although this is generally not required [4,58].

The DMD algorithm constructs the leading eigendecomposition of the best-fit operator A , chosen to minimize $\sum_k \|x_{k+1} - Ax_k\|_2$ over the $k = 0, 2, 3, \dots, m - 1$ snapshots, so that $X' \approx AX$. Computationally, the matrix A is obtained as

$$A = X' X^\dagger \tag{4}$$

where $A \in \mathbb{R}^{n \times n}$, and X^\dagger is the Moore–Penrose pseudoinverse of X . The dominant eigenvectors, corresponding to the least damped eigenvalues of A , are the dynamic modes Φ , and the associated eigenvalues determine how these modes behave in time.

In practice, the high-dimensional A is not computed directly. Instead, the SVD can be used to first project to a low-rank subspace and then compute the matrix $\tilde{A} = U^* A U$, which has many of the same eigenvalues as A . This provides an efficient algorithm whose computational expense is bounded by the rank r of the data. The exact DMD algorithm of Tu et al. [4] is shown in Algorithm 1. Using a variable projection optimization scheme, the exact DMD method can be modified to handle arbitrary temporal spacing between snapshots. It further produces a more robust or optimal DMD approximation for noisy data [58].

A. Dynamic Mode Decomposition with Control

The dynamic mode decomposition with control (DMDc) method is a critically enabling extension of DMD [28]. DMDc disambiguates between the underlying dynamics and the effects of actuation, modifying the basic assumption of DMD to include the effect of inputs $u_k \in \mathbb{R}^q$

$$x_{k+1} = Ax_k + Bu_k \tag{5}$$

where $B \in \mathbb{R}^{n \times q}$. The matrix form of the actuation is

$$Y = \begin{bmatrix} | & | & & | \\ u_0 & u_1 & \dots & u_{m-1} \\ | & | & & | \end{bmatrix} \tag{6}$$

The system can be written in matrix form as

$$X' = AX + BY \tag{7}$$

where each column u_k of the input snapshot matrix Y is the input at each snapshot in Eq. (6), and the data matrices X, X' are formulated in the same way as in Eq. (2).

A least-squares regression algorithm can once again be used to determine the matrix A and its associated DMD modes and eigenvalues. Two distinguishing cases can be considered: when B is known and when B is unknown. When the input matrix B is known or can be well-estimated, the output is a simple linear combination of states and inputs. The DMD modes can then be obtained following the procedure of the exact DMD algorithm 1, with X' replaced with $X' - BY$. For an unknown B , it is possible to compute the DMD modes and an approximation to the matrix B via regression [28]. For this case, an augmented matrix containing both the state and input snapshots is constructed:

$$\Omega = \begin{bmatrix} X \\ Y \end{bmatrix} \tag{8}$$

along with an augmented matrix containing the two unknown system matrices:

$$G = [A \quad B] \tag{9}$$

The regression problem is then formulated as

$$X' = [A \quad B] \begin{bmatrix} X \\ Y \end{bmatrix} = G\Omega \tag{10}$$

where $\Omega \in \mathbb{R}^{(n+q) \times m}$ is the combination of the state and control snapshots. The DMDc algorithm with an unknown B is shown in Algorithm 2. Importantly, the SVD now constructs a low-rank representation of the state and input variables, both of which are used to produce approximations of the matrices A and B through low-rank structures. In all of the following examples, we choose the

Algorithm 2 DMD with control [28]

Input: Data matrices X, X' , input snapshot matrix Y , target rank r of X or X' and \tilde{r} of Ω .
Optional: Actuation matrix B .
Output: Spectrum Λ , modes Φ , [and actuation matrix \hat{B}]. ▷ Optional outputs in brackets [-].
1: **procedure** DMDc($X, X', Y, r, \tilde{r}, [B]$)
2: **if** B is known **then**
3: $[\Lambda, \Phi] \leftarrow \text{DMD}(X, X' - BY, r)$ ▷ Perform DMD (Algorithm 1) adjusted
4: ▷ for known actuation.
5: **else**
6: $\Omega \leftarrow \begin{bmatrix} X \\ Y \end{bmatrix}$ ▷ Matrix of the state and input snapshots.
7: $[\tilde{U}, \tilde{\Sigma}, \tilde{V}] \leftarrow \text{SVD}(\Omega, \tilde{r})$ ▷ Truncated \tilde{r} -rank SVD of Ω .
8: $\tilde{U}_1, \tilde{U}_2 \leftarrow \tilde{U}$ ▷ Split \tilde{U} into two components.
9: $[\hat{U}, \hat{\Sigma}, \hat{V}] \leftarrow \text{SVD}(X', r)$ ▷ Truncated r -rank SVD of X' .
10: $\tilde{A} \leftarrow \hat{U}^* X' \tilde{V} \hat{\Sigma}^{-1} \tilde{U}_1^*$ ▷ Low-rank approximation of \tilde{A} .
11: $\tilde{B} \leftarrow \hat{U}^* X' \tilde{V} \hat{\Sigma}^{-1} \tilde{U}_2^*$ ▷ Estimate reduced actuation matrix \tilde{B} .
12: $\hat{B} \leftarrow X' \tilde{V} \hat{\Sigma}^{-1} \tilde{U}_2^*$ ▷ Estimate actuation matrix \hat{B} .
13: $[W, \Lambda] \leftarrow \text{EIG}(\tilde{A})$ ▷ Eigendecomposition of \tilde{A} .
14: $\Phi \leftarrow X' \tilde{V} \hat{\Sigma}^{-1} \hat{U}_1^* W$ ▷ DMD modes of \tilde{A} .
15: **end if**
16: **end procedure**
Note that if $\lambda_i = 0$, then $\phi_i = \tilde{U}_1 \tilde{U}_1^* \hat{U}_1 w_i$ for step 13.

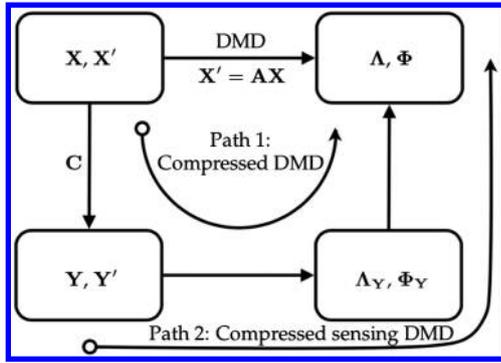


Fig. 1 Schematic of DMD and compressed DMD. Path 1 shows compressed DMD, and path 2 shows compressed sensing DMD [41].

rank $\tilde{r} = r - q$. Much like the DMD algorithm, DMDc relies on least-squares regression of the data to build a linear model A of the state dynamics that is disambiguated from the discovered actuation matrix B . In general, DMDc may be applied to systems with high-dimensional inputs, where q is large. However, it may be challenging to fully characterize the relevant subspaces without collecting significant amounts of input–output data.

B. Compressed Sensing and Dynamic Mode Decomposition

Another innovation of the DMD algorithm addresses limitations on measurement and acquisition of a dynamical system. Such limited data acquisition is often imposed by physical constraints, such as data-transfer bandwidth in particle image velocimetry (PIV), or the costs of sensors. Given the low-rank nature of the spatiotemporal structures exhibited in many complex systems, we can use ideas from compressed sensing [37–39] to reconstruct the full high-dimensional state x from a small number of measurements.

Compressive DMD (cDMD) develops a strategy for computing the dynamic mode decomposition from compressed or subsampled data [41].

Consider compressed or subsampled data Y given by

$$Y = CX \quad (11)$$

where C is a measurement matrix. There are two key strategies to cDMD, as illustrated in Fig. 1.

First, it is possible to reconstruct full-state DMD modes from heavily subsampled or compressed data using compressed sensing. This is called compressed sensing DMD, and it is appropriate to use when access to the full state space is not possible due to constraints on physical measurements. Second, if full-state snapshots are available, it is possible to first compress the data, perform DMD, and then reconstruct by taking a linear combination of the snapshot data, determined by the DMD on compressed data. This is called

compressed DMD. In this case, it is assumed that one has access to the full high-dimensional state-space data. The theory for either of these methods relies on relationships between DMD on full-state and compressed data. Importantly, when data and modes are sparse in some transform basis, there is an invariance of DMD to measurement matrices that satisfy the restricted isometry property (RIP) from compressed sensing.

In addition to the data matrices X and X' , it is also now required to specify a measurements matrix C . These three matrices together are required to execute Algorithm 3. In practice, we often consider point measurements so that the rows of C are rows of the identity matrix. For DMD modes that are global in nature, point measurements naturally satisfy the RIP property because they are incoherent with respect to the global modes. The success of the method, both the compressive-sampling DMD and the cDMD, has been demonstrated by Brunton et al. [41] to be an effective strategy for subsampling of data and the reconstruction of DMD modes and eigenvalues.

III. Compressive System Identification

A major benefit of dynamic mode decomposition is that it provides a physically interpretable and highly extensible linear model framework, which enables the incorporation of actuation inputs and sparse output measurements. When combined, these innovations result in the compressive DMD with control (cDMDc) architecture for compressive system identification, where low-order models are identified from input–output measurements. In contrast to traditional system identification, the reduced states of the cDMDc models may be used to reconstruct the high-dimensional state space via compressed sensing, adding physical interpretability to the states of the reduced models. Thus, cDMDc relies on the existence of a few dominant coherent patterns, which in turn facilitates sparse measurements.

We now consider a general input–output system with high-dimensional state x :

$$x_{k+1} = Ax_k + Bu_k \quad (12a)$$

$$y_k = Cx_k \quad (12b)$$

As in DMD, the goal is to obtain the leading eigendecomposition of A , resulting in a low-order model in terms of a few DMD modes. However, now we must account for limited measurements in the output y and disambiguate internal state dynamics from the effect of actuation u . Writing Eq. (12) in terms of data matrices yields

$$X' = AX + BY \quad (13a)$$

$$Y = CX \quad (13b)$$

Algorithm 3 Compressive DMD [41]

Input: Measurements Y, Y' , measurement matrix C , sparsifying basis Ψ , and target rank r .

Optional: X, X' .

Output: cDMD spectrum Λ and modes $\hat{\Phi}$.

1: **procedure** cDMD($Y, Y', C, r, [X, X']$).

2: $[U_Y, \Sigma_Y, V_Y] \leftarrow \text{SVD}(Y, r)$

3: $A_Y \leftarrow U_Y^* Y' V_Y \Sigma_Y^{-1}$

4: $[W_Y, \Lambda_Y] \leftarrow \text{EIG}(A_Y)$

5: **if** X is known **then**

6: $\hat{\Phi} \leftarrow X' V_Y \Sigma_Y^{-1} W_Y$

7: **else**

8: $\Phi_Y \leftarrow Y' V_Y \Sigma_Y^{-1} W_Y$

9: $\Phi_S \leftarrow \text{Compressed sensing}(\Phi_Y, C, \Psi)$

10: $\hat{\Phi} \leftarrow \Psi \Phi_S$

11: **end if**

12: **end procedure**

Note that if $\lambda_i = 0$, then $\phi_i = U w_{Y,i}$, $\phi_{Y,i} = U_Y w_{Y,i}$ for steps 6 and 8.

▷ Truncated r -rank SVD of Y .

▷ Low-rank approximation of A_Y .

▷ Eigendecomposition of A_Y .

▷ Perform compressed DMD.

▷ Estimate DMD modes of A .

▷ Perform compressed sensing DMD.

▷ DMD modes of A_Y .

▷ Perform l_1 minimization on ϕ_Y , to solve for ϕ_S .

▷ Estimate DMD modes of A .

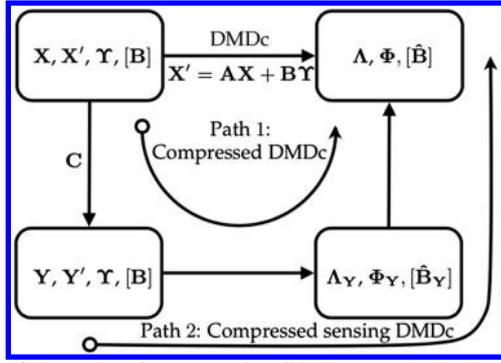


Fig. 2 Schematic of compressive DMD control. Path 1 shows compressed DMDc, and path 2 shows compressed sensing DMDc.

Under certain conditions, it is possible to apply DMDc to the compressed data Y and then recover full-state DMD modes via compressed sensing. As in the compressive DMD algorithm [41], if full-state data X are available, significant computational savings may be attained by compressing the data and working in the compressed subspace. If full-state data are unavailable, it may still be possible to reconstruct full-state modes via convex optimization, exploiting sparsity of the modes in a generic basis, such as Fourier or wavelets. The cDMDc framework is shown in Fig. 2 and is described in Algorithm 4.

We now prove that DMD eigenvectors of the full data X map into DMD eigenvectors of the compressed data Y , when multiplied by the compression operator C . This section relies on notation developed previously, which is consolidated in the Nomenclature. Matrices with a subscript Y are computed on compressed data, and matrices with a tilde are computed from the SVD of the augmented matrix Ω .

Assumption 1: The measurement matrix C preserves the temporal information in Ω so that $\tilde{V}_Y \tilde{V}_Y^* \tilde{V} = \tilde{V}$. This requires the columns of \tilde{V} to be in the column space of \tilde{V}_Y and will only be approximately satisfied with measurement noise.

Assumption 2: The columns of the full-state output matrix X' are in the subspace of the upper left singular vectors \tilde{U}_1 of the full-state augmented matrix Ω , so that $\tilde{U}_1 \tilde{U}_1^* X' \approx X'$.

Lemma 1: If Assumption 1 holds, we have an identity $\tilde{V} \tilde{\Sigma}^{-1} = \tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,1}^* C \tilde{U}_1$, similar to that derived in [41].

Proof:

$$\begin{aligned} Y &= CX \\ \tilde{U}_{Y,1} \tilde{\Sigma}_Y \tilde{V}_Y^* &= C \tilde{U}_1 \tilde{\Sigma} \tilde{V}^* \\ \tilde{V}_Y^* \tilde{V} \tilde{\Sigma}^{-1} &= \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,1}^* C \tilde{U}_1 \\ \tilde{V}_Y \tilde{V}_Y^* \tilde{V} \tilde{\Sigma}^{-1} &= \tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,1}^* C \tilde{U}_1 \\ \tilde{V} \tilde{\Sigma}^{-1} &= \tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,1}^* C \tilde{U}_1 \end{aligned}$$

The next theorem uses the following definitions of A and A_Y from the DMDc and cDMDc algorithms:

$$\begin{aligned} A &= X' \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}_1^* \\ A_Y &= Y' \tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,1}^* \end{aligned}$$

Theorem 1: If Assumption 1 and 2 hold, then

$$CAX' = A_Y CX' \quad (14)$$

Proof: Using Lemma 1, we have

$$\begin{aligned} CAX' &= C(X' \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}_1^*) X' \\ &= Y' (\tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,1}^* C \tilde{U}_1) \tilde{U}_1^* X' \\ &= A_Y C \tilde{U}_1 \tilde{U}_1^* X' \\ &= A_Y CX' \end{aligned}$$

Thus, the compression matrix C commutes with the dynamics in A , when applied to data X' . We use this to obtain the central result: compressed dynamic modes of the full data are dynamic modes of the compressed data.

Theorem 2: If Assumptions 1 and 2 hold, then compressing a full-state DMDc eigenvector ϕ yields a DMDc eigenvector of the compressed data with the same eigenvalue:

$$A_Y C \phi = \lambda C \phi \quad (15)$$

Algorithm 4 Compressive DMD with control

Input: Measurement snapshot matrices Y, Y' , sensing matrix C , input snapshot matrix Y , and target rank r, \tilde{r} .

Optional: Full-state data matrices X, X' and actuation matrix B .

Output: cDMDc spectrum Λ and modes $\hat{\Phi}, [\hat{B}]$.

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1: procedure cDMDc( $Y, Y', C, r, \tilde{r}, [X, X', B]$ )
2:   if  $X$  is known then
3:     if  $B$  is known then
4:        $\Lambda, \hat{\Phi} \leftarrow$  cDMD( $Y, Y' - CBY, C, r, X, X'$ )  $\triangleright$  Perform compressed DMD.
5:     else
6:        $\Lambda_Y, \hat{U}_Y, \tilde{U}_{Y,1}, \tilde{U}_{Y,2}, \tilde{\Sigma}_Y, \tilde{V}_Y, W_Y$ 
7:        $\leftarrow$  DMDc( $Y, Y', Y, r, \tilde{r}$ )  $\triangleright$  Perform DMDc.
8:        $\hat{\Phi} \leftarrow X' \tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,1}^* \tilde{U}_Y W_Y$   $\triangleright$  Estimate DMD modes of  $A$ .
9:        $\hat{B} \leftarrow X' \tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,2}^*$   $\triangleright$  Estimate actuation matrix  $\hat{B}$ .
10:    end if
11:  else
12:    if  $B$  is known then
13:       $\Lambda, \hat{\Phi} \leftarrow$  cDMD( $Y, Y' - CBY, C, r$ )  $\triangleright$  Perform compressed sensing DMD.
14:    else
15:       $\Lambda_Y, \Phi_Y, \hat{B}_Y \leftarrow$  DMDc( $Y, Y' - Y, r, \tilde{r}$ )  $\triangleright$  Perform DMDc.
16:       $\hat{\Phi} \leftarrow$  Compressed Sensing( $\Phi_Y, \Psi$ )  $\triangleright$  Estimate DMD modes of  $A$ .
17:       $\hat{B} \leftarrow$  Compressed Sensing( $\hat{B}_Y, \Psi$ )  $\triangleright$  Estimate actuation matrix  $\hat{B}$ .
18:    end if
19:  end if
20: end procedure

```

Note that if $\lambda_i = 0$, then $\phi_i = \tilde{U} \tilde{U}_{Y,1}^* \tilde{U}_Y w_{Y,i}$ for step 7.

Proof:

$$\begin{aligned} A_Y C \phi &= A_Y C X' \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}_1^* \hat{U} w \\ &= C A X' \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}_1^* \hat{U} w \\ &= C A \phi \\ &= \lambda C \phi \end{aligned}$$

If C is chosen poorly so that ϕ is in its null space, then Theorem 2 applies trivially. This theorem does not guarantee that every eigenvector of A_Y is a compressed eigenvector of A , although under reasonable assumptions the dominant eigenvalues of A_Y will approximate those of A , as shown in [41]. We then have

$$\phi_Y = C \phi_X = C \Psi \phi_S \quad (16)$$

where ϕ_S is the sparse representation of ϕ_X in a universal basis Ψ , such as a Fourier or wavelet basis. Thus, it is possible to recover ϕ_S , and hence ϕ_X , from compressed DMD modes ϕ_Y , given enough incoherent measurements [41].

Compressed recovery of the actuation matrix: We now establish a similar relationship between the full-state actuation matrix B and the compressed matrix B_Y .

Assumption 3: The columns of $\tilde{U}_{Y,2}$ are spanned by those of \tilde{U}_2 , so $\tilde{U}_2 \tilde{U}_2^* \tilde{U}_{Y,2} = \tilde{U}_{Y,2}$ and $\tilde{U}_{Y,2}^* \tilde{U}_2 \tilde{U}_2^* = \tilde{U}_{Y,2}^*$.

Lemma 2: We have an identity $\tilde{V} \tilde{\Sigma}^{-1} = \tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,2}^* \tilde{U}_2$.

Proof: Expanding \mathbf{Y} in the SVD bases of Ω_Y and Ω , we find

$$\begin{aligned} \tilde{U}_{Y,2} \tilde{\Sigma}_Y \tilde{V}_Y^* &= \tilde{U}_2 \tilde{\Sigma} \tilde{V}^* \\ \tilde{V}_Y^* \tilde{V} \tilde{\Sigma}^{-1} &= \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,2}^* \tilde{U}_2 \\ \tilde{V}_Y \tilde{V}_Y^* \tilde{V} \tilde{\Sigma}^{-1} &= \tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,2}^* \tilde{U}_2 \\ \tilde{V} \tilde{\Sigma}^{-1} &= \tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,2}^* \tilde{U}_2 \end{aligned}$$

The next theorem uses the following definitions of B and B_Y from the DMDc and cDMDc algorithms:

$$\begin{aligned} B &= X' \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}_2^* \\ B_Y &= Y' \tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,2}^*. \end{aligned}$$

Theorem 3: If Assumptions 1 and 3 hold, then

$$C B = B_Y \quad (17)$$

Proof: Using Lemma 2, we have

$$\begin{aligned} C B &= C X' \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}_2^* \\ &= Y' \tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,2}^* \tilde{U}_2 \tilde{U}_2^* \\ &= Y' \tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,2}^* \\ &= B_Y \end{aligned}$$

Therefore, we can first compute the DMD modes Φ_Y and the compressed actuation matrix B_Y and then reconstruct the full-state Φ and B through compressed sensing.

A. Relationship to System Identification

The result from Theorem 1 also carries over to general impulse response parameters in the following theorem. These impulse response parameters, also known as Markov parameters, are used extensively in system identification, for example to construct Hankel matrices in the eigensystem realization algorithm (ERA) [27].

Theorem 4: If Assumptions 1 and 2 hold, then

$$C A B = A_Y C B \quad (18)$$

Proof: Using Lemma 1, we have

$$\begin{aligned} C A B &= C (X' \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}_1^*) B \\ &= Y' (\tilde{V}_Y \tilde{\Sigma}_Y^{-1} \tilde{U}_{Y,1}^* C \tilde{U}_1) \tilde{U}_1^* B \\ &= A_Y C \tilde{U}_1 \tilde{U}_1^* X' \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}_2^* \\ &= A_Y C X' \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}_2^* \\ &= A_Y C B \end{aligned}$$

Corollary 1: Theorem 4 can be expanded to further steps in dynamics, for $k \in \mathbb{Z}^+$, such as

$$C A^k B = A_Y^k C B = A_Y^k B_Y \quad (19)$$

This theorem and corollary establish a surprising result; under certain conditions, the iterative dynamics in the impulse response parameters commute with the measurement matrix. Impulse response parameters are a cornerstone of subspace system identification methods, such as ERA, and the preceding results simplify the dynamics.

Corollary 1 also has implications for the controllability of the projected and full-state systems. Given the controllability matrix

$$C = \begin{bmatrix} B & A B & \cdots & A^{n-1} B \end{bmatrix} \quad (20)$$

we have the following relationship:

$$C C = C \begin{bmatrix} B & A B & \cdots & A^{n-1} B \end{bmatrix} \quad (21a)$$

$$= \begin{bmatrix} B_Y & A_Y B_Y & \cdots & A_Y^{n-1} B_Y \end{bmatrix} = C_Y \quad (21b)$$

Therefore, if the full-state systems is controllable (i.e., the controllability matrix C is full rank), and the compressed measurement matrix C is not in the null space of Eq. (20), then the compressed system is also controllable. The controllability is preserved after compression under certain conditions. The observability property regarding the compressed sensing framework was discussed in [59].

B. Computational Considerations

Similar to cDMD, there are two main paths presented in compressive DMD with control, depending on the availability of the full-state data matrix X . Specifically, we refer to the first path as compressed DMDc (path 1 in Fig. 2) if X is known and the second path as compressed sensing DMDc (path 2 in Fig. 2) if X is only partially known due to the lack of full state measurements.

1) If the full-state measurements X are available, compressed DMDc is advantageous because the expensive calculations are performed on the compressed data Y , and the full-state modes are obtained by linearly combining the snapshots of X .

2) Without access to full-state measurements X , compressed sensing DMDc extracts the inherent dynamics from subsampled/compressed data in the matrix Y , disambiguating the effect of actuation. Full-state modes may then be recovered, under the standard conditions of compressed sensing.

In addition, two approaches are considered in each path, considering the prior knowledge of B . Generally, the cDMDc algorithm can be simplified as a corrected cDMD algorithm if B is known. Otherwise, DMDc is used to extract the underlying dynamics from the compressed states Y , Y' , and then the modes Φ and actuation matrix B are reconstructed either by projecting onto the full states X' or through compressed sensing.

In Algorithm 4, these four scenarios are discussed. When both X and B are known, the spectrum Λ_Y and modes Φ are obtained by performing compressed DMD on the pre-compressed data Y and the shifted matrix correcting for the effect of control $Y' - C B Y$. When X

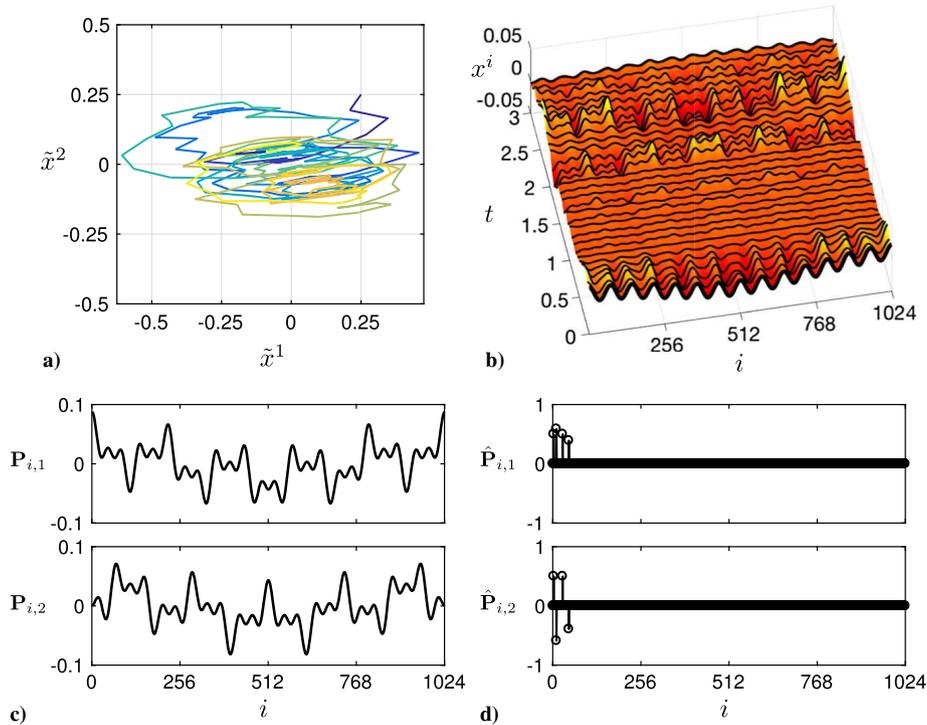


Fig. 3 Two-dimensional system: a) phase portrait (color denotes progression of time), b) x - t diagram of high-dimensional system, c) two orthogonal modes of P , and d) DCT coefficients of the modes.

is known and B is unknown, DMDC is computed on Y , Y' and the dynamic modes Φ and actuation matrix B are reconstructed as a linear combination of the full-state data X' . When X is only partially known and B is known, compressed sensing DMDC is performed on Y and $Y' - CBY$ and the full-state modes Φ are reconstructed from the compressed Φ_Y . When both X and B are unavailable, DMDC is computed on Y , Y' and the dynamic modes Φ and actuation matrix B are reconstructed using compressed sensing.

In the compressed DMDC case, the computational cost is dominated by the singular-value decomposition. When X is known, the computational cost is reduced from $\mathcal{O}(nm^2)$ in DMDC to $\mathcal{O}(\min(pm^2, mp^2))$ in cDMDC. Thus, the cost reduction is roughly proportional to the compression ratio when $p > m$ and is even more significant when $p < m$. When X is unavailable, compressed sensing DMDC is necessary, and the CoSaMP recovery has the added time complexity of $\mathcal{O}(n \log^2 n)$ [60]. However, in this case, computational cost is not the main driving factor.

IV. Results

In this section, we present two numerical experiments that illustrate compressive DMD with control. In the first example, we investigate a spatially high-dimensional system that is lifted from a two-dimensional system with known dynamics and random control input. In the second example, we model the vorticity field of a fluid flow downstream of a pitching plate at low Reynolds number. In both examples, we investigate the effectiveness of different random measurement matrices, compression ratios, and actuation input vectors.

A. Stochastically Forced Linear System

This experiment is designed to test the compressive DMD with control framework on an example where the low-rank dynamics are known. Thus, it is possible to directly compare the true eigenvalue spectrum and spatiotemporal modes with those obtained via compressed DMDC and compressed sensing DMDC.

The state matrix A and input matrix B are designed to yield a stable, controllable system:

$$\begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{bmatrix}_{k+1} = \underbrace{\begin{bmatrix} 0.9 & 0.2 \\ -0.1 & 0.9 \end{bmatrix}}_{\hat{A}} \begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{bmatrix}_k + \underbrace{\begin{bmatrix} 0.1 \\ 0.01 \end{bmatrix}}_{\hat{B}} u_k \quad (22)$$

The dynamics are excited via Gaussian random input excitation in u_k , starting from an initial condition $x_0 = [0.25 \ 0.25]^T$. The system is integrated from $t = 0$ to $t = 30$ with a time step of $\Delta t = 0.1$, resulting in 301 snapshots. A sample trajectory of the low-dimensional system is shown in Fig. 3a.

To inflate the state dimension, we associate each of the two states \tilde{x}^1 and \tilde{x}^2 with a high-dimensional mode that is sparse in the spatial wavenumber domain. These modes, shown in Fig. 3c, are given by the columns of $P \in \mathbb{R}^{1024 \times 2}$, where each column is constructed to have $K = 4$ non-zero elements in the discrete cosine transform (DCT) basis. The nonzero DCT coefficients of each mode have the same wave number with different magnitudes, as shown in Fig. 3d. Thus, it is possible to use the DCT matrix as the sparsifying basis for compressed sensing. With the spatial modes in P , it is possible to lift the low-dimensional state \tilde{x}_k to a high-dimensional state $x_k = P\tilde{x}_k$. The lifted state is shown in Fig. 3 for the first 31 snapshots, from $t = 0$ to $t = 3$. Note that the high-dimensional actuation vector B is chosen to be in the span of the columns of P , i.e. $B = P\hat{B}$, so that the actuation only excites low-dimensional dynamics; other types of actuation will be examined further in Fig. 4.

To investigate the proposed cDMDC algorithm, we now consider compressed measurements y given by Eq. (12b). Specifically, the compression matrix C can be built with entries drawn from uniform (C -type 1), Gaussian (C -type 2) or Bernoulli (C -type 3) distributions. Fig. 5 shows the DMDC mode reconstruction using compressed DMDC (Path 1 in Fig. 2) and compressed sensing DMDC (Path 2 in Fig. 2) for $p = 128$ Gaussian measurements. In both cases, it is assumed that the high-dimensional actuation input vector B is known, and the reconstructed modes faithfully reproduce the true coherent structures of the underlying system (i.e., the DMD modes are a linear combination of the columns of P). The results remain unchanged when B is unknown and must also be reconstructed.

Table 1 shows the error in the reconstructed eigenfunctions Φ and estimated actuation vector \hat{B} (when unknown) for DMDC, compressed DMDC, and compressed sensing DMDC, compared against the true

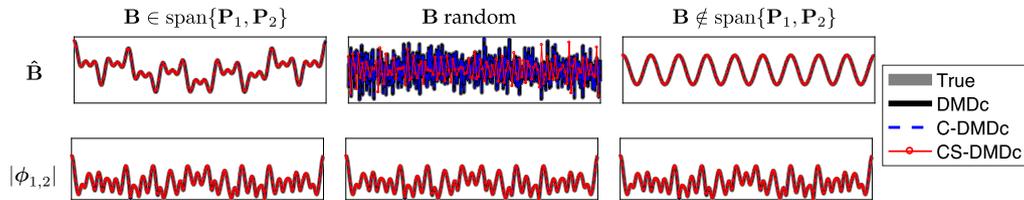


Fig. 4 Estimated $\hat{\mathbf{B}}$ and mode magnitudes for different \mathbf{B} in DMDc, compressed DMDc, and compressed sensing DMDc. For compressed DMDc, the same number of measurements is employed as for Fig. 5.

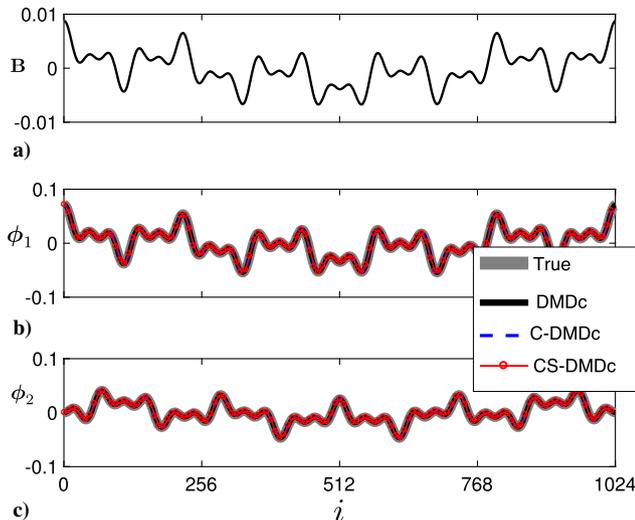


Fig. 5 Reconstruction of a) actuation vector \mathbf{B} , and b–c) modes from DMDc, compressed DMDc, and compressed sensing DMDc. For compressive DMD, 128 Gaussian random measurements are collected from 1024 state dimensions.

values. In addition, we investigate the effect of different sensing strategies discussed above. In all cases, the reconstruction error is small, as no noise is added to the simulated data.

Compressed sensing DMDc is able to uncover the underlying dynamics and spatiotemporal modes from noiseless subsampled data, relaxing the requirement of high-dimensional measurements. However, in realistic experimental conditions, measurement noise will always be present and is known to affect DMD computations. Figure 6 shows the performance of DMDc and compressed DMDc for varying levels of measurement noise, averaged over 100 different noise realizations in each case. Note that compressed sensing DMDc and compressed DMDc have identical eigenvalues because both methods compute the DMD spectrum from the same compressed data. For moderate noise levels, the compressed DMDc algorithm provides reasonably accurate eigenvalues, although they are less accurate than those predicted by DMDc. As the noise intensity is increased to greater than 50%, both the DMDc and cDMDc

eigenvalues begin to deviate, and in some cases, complex conjugate eigenvalues are miscomputed as purely real eigenvalues. Although the effect of noise is exacerbated by compression, sensitivity of DMD eigenvalues with measurement noise is a known issue and has been extensively studied and characterized [61–63]. There are a number of algorithmic extensions that improve the eigenvalue prediction with noise, including using the total least squares [62], a forward-backward symmetrizing algorithm [63], subspace DMD [64], Bayesian DMD [65], or an optimal DMD based on variable projection methods [58]. Each of these methods may be effectively combined with the proposed cDMDc architecture to yield more accurate eigenvalues.

Finally, we investigate the performance of cDMDc for different choices of the actuation vector \mathbf{B} in Fig. 4. In addition to the cases presented so far, where \mathbf{B} is in the subspace of \mathbf{P} , we now explore scenarios when \mathbf{B} is randomly generated or in the complementary subspace of \mathbf{P} . For all types of actuation \mathbf{B} , and regardless of whether or not \mathbf{B} is known, compressed DMDc and compressed sensing DMDc both accurately identify the true modes ϕ_1 and ϕ_2 , which is also consistent with DMDc. When \mathbf{B} is unknown, cDMDc and csDMDc both accurately identify \mathbf{B} , regardless of the subspace it belongs to, as long as the columns of \mathbf{B} are sparse. The algorithms do not accurately capture the \mathbf{B} matrix when it is not sparse (i.e., random actuation vector); however, in this case, DMDc also misidentifies the actuation vector.

B. Fluid Flow Past a Pitching Plate

In the second example, we apply the proposed compressive DMD with control algorithm to model the vorticity field downstream of a pitching plate at Reynolds number $Re = 100$. This pitching airfoil has been studied previously in the context of reduced-order models for flow control [66–70].

The flow is simulated using the immersed boundary projection method** [66,67] with a grid resolution of 799×159 on a domain of size 10×2 , nondimensionalized by the plate length L . The flow physics may be more accurately resolved and boundary effects minimized by using nested grids, as in [67]. The flow is simulated with a time step of $\Delta t = 0.01$ dimensionless convective time units, nondimensionalized by the length L and freestream velocity U . Two hundred fifty-one snapshots are sampled at a rate of $\Delta t = 0.1$. In this example, the airfoil is rapidly pitched up and down between ± 5 deg at irregular intervals in time, using the canonical pitch maneuver described in [71]. Six rapid pitch maneuvers are performed, and then the data are symmetrized by concatenating a mirror image of these 251 snapshots with the opposite signed vorticity, in an attempt to identify unbiased modes. Figure 7 illustrates the vorticity field at different times $t = 0, 11.5, 21.5$. We focus on a downstream measurement window of the size 399×141 to mimic a PIV window. The actuation input takes 4.6 convective time units to reach the observation window, and the actuation input, used as input for cDMDc and DMDc, is shifted accordingly. We choose small amplitude pitching motions because they have been shown to be well approximated with linear models [68], which make it appropriate for DMDc and cDMDc.

**Data available online at <https://github.com/cwrowley/ibpm> [retrieved 20 June 2017].

Table 1 Normalized error of $\|\Phi - \hat{\Phi}\|_F$ and $\|\mathbf{B} - \hat{\mathbf{B}}\|_2$ using DMDc methods for: uniform random projections (C-1), Gaussian random projections (C-2), and singlepixel measurements (C-3)

C type	DMDc	c-DMDc	cs-DMDc
<i>\mathbf{B} known</i>			
1	3.631e-13	3.627e-13	3.627e-13
2	3.631e-13	3.443e-13	3.443e-13
3	3.631e-13	4.210e-13	4.210e-13
<i>\mathbf{B} unknown</i>			
1	4.481e-13, 1.758e-16	5.396e-13, 4.840e-17	5.926e-13, 9.558e-17
2	4.481e-13, 1.758e-16	4.159e-13, 4.731e-17	3.872e-13, 2.845e-16
3	4.481e-13, 1.758e-16	4.322e-13, 3.022e-16	5.691e-13, 2.785e-16

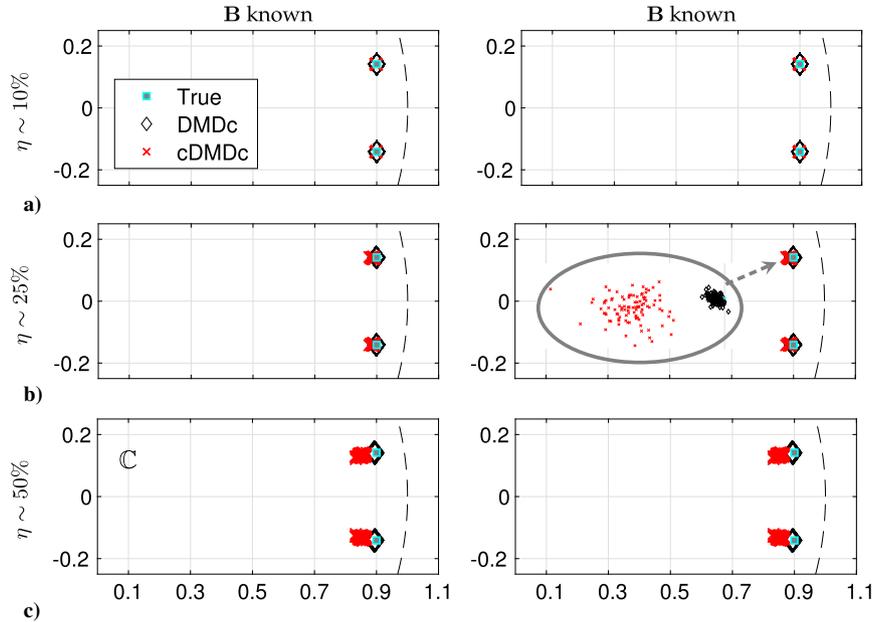


Fig. 6 Noise dependency of DMD methods based on 100 noise realizations. Rows correspond to noise levels $\eta \in \{0.1, 0.25, 0.5\}$ with $\sigma_{\text{noise}} = \eta \max(\sigma_i)$, where σ_i denotes the measurement standard deviation.

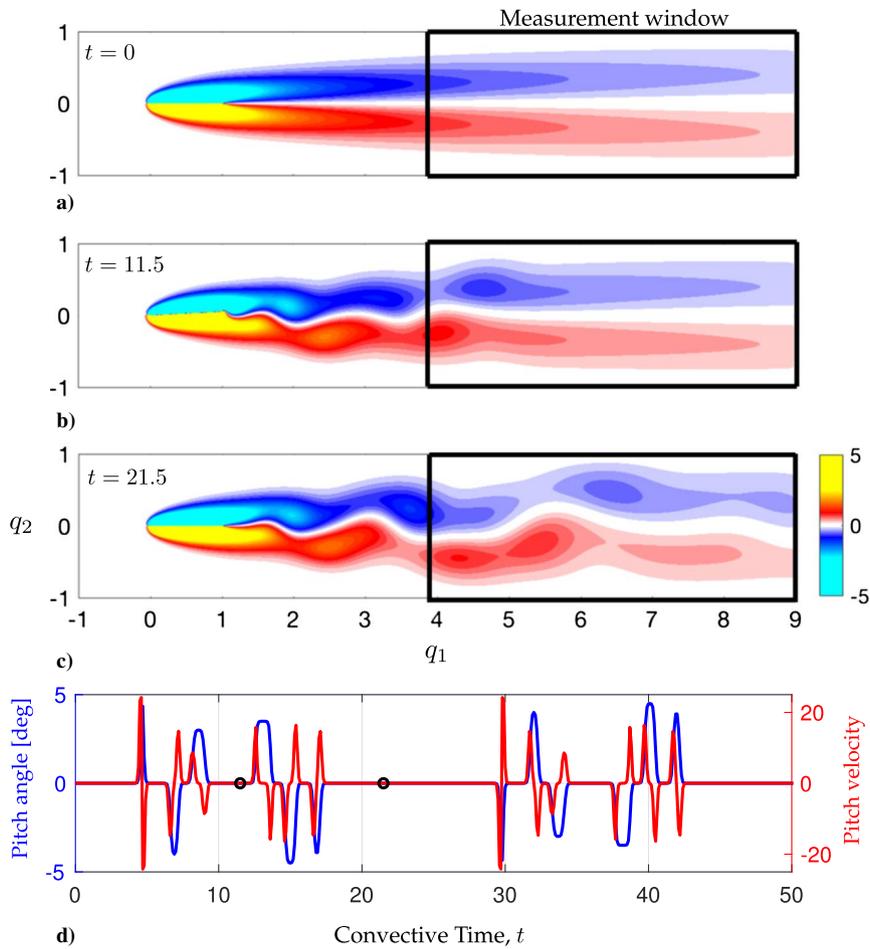


Fig. 7 Pitching airfoil: a–c) instantaneous vorticity at times $t = 0, 11.5, 21.5$ and showing the measurement window, and d) time series of actuation (pitching of the plate).

The eigenvalues of \tilde{A} and \tilde{A}_γ given by DMDc and cDMDc are shown in Fig. 8. The cloud of cDMDc eigenvalues is generated from an ensemble of 50 realizations using 10% compressed measurements (i.e., $p = 0.1n$) based on Gaussian random projections. We use $r = 9$ in this study to identify the spatiotemporal coherent DMD

modes, and they capture 96% of the total energy, based on the singular values. These nine modes are ordered by their decay rate (i.e., the real part) and the higher-order modes have larger frequencies of oscillation. The discrete-time eigenvalues are located close to the unit circle, and they are slightly stable, as seen in the continuous-time

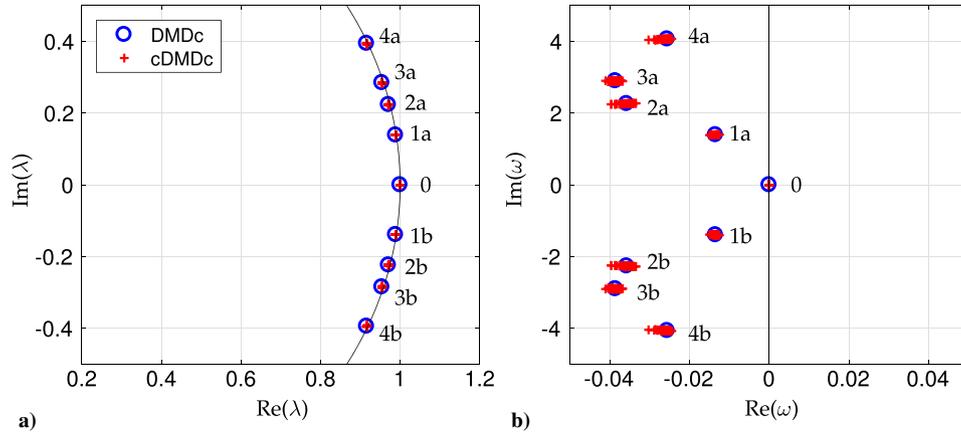


Fig. 8 Comparison of the spectrum in a) discrete time, and b) continuous time, obtained using DMDc and cDMDc with 10% Gaussian random measurements. The first $r = 9$ modes are shown.

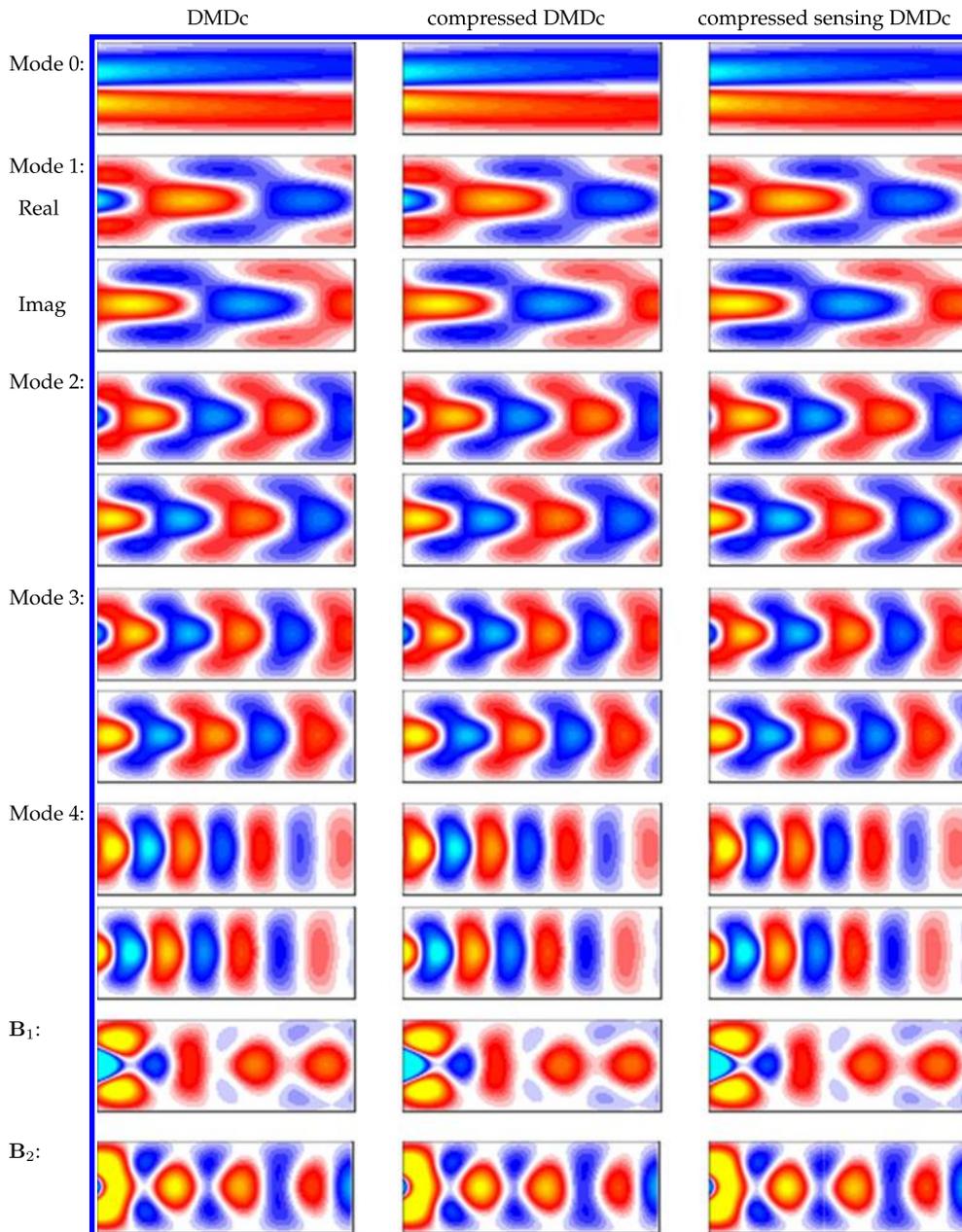


Fig. 9 Estimation of real and imaginary mode pairs and B using DMDc methods using p Gaussian random projections, where $p/n = 0.1$ (10% compression ratio). Single-pixel measurements are similar (not shown).

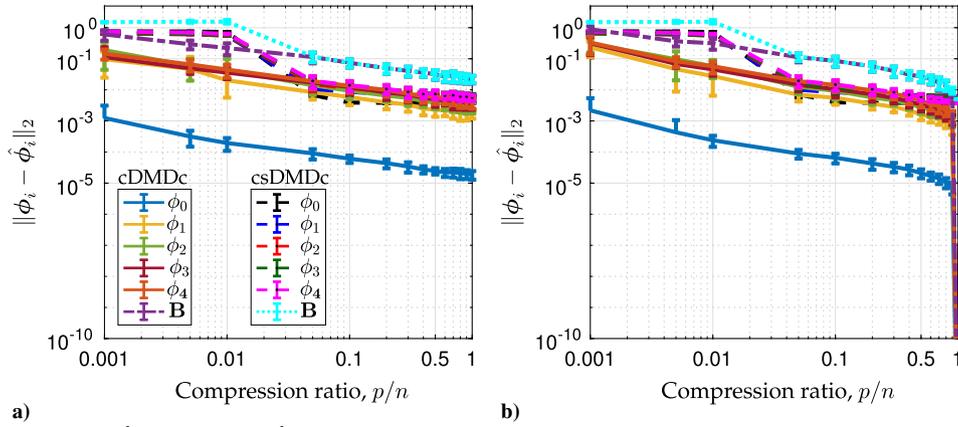


Fig. 10 Error of estimated modes $\hat{\phi}_i$ and actuation \hat{B} for increasing compression ratio using compressed and compressed sensing DMDC for a) Gaussian random projections, and b) single-pixel measurements.

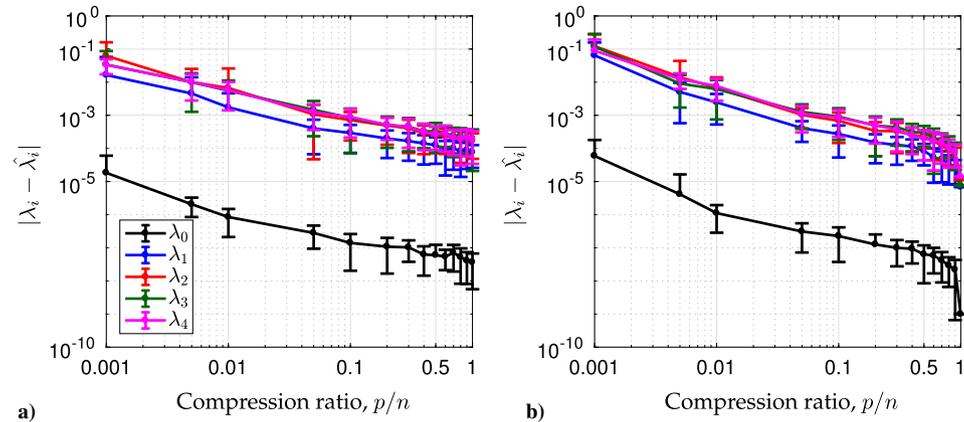


Fig. 11 Error of eigenvalues for increasing compression ratio using a) Gaussian random projections, and b) single-pixel measurements. The DMDC eigenvalues are used as the reference λ_i .

plot. The DMDC eigenvalues appear to have a decay rate that is too small, although the dynamics of transients are captured quite well. Nevertheless, the cDMDC eigenvalues agree well with the DMDC eigenvalues, which is the goal.

In Fig. 9, we present the spatial structure of the DMDC mode pairs and the \mathbf{B} matrix. Similar DMDC modes have been observed in the flow past a cylinder [3]. As the temporal frequency associated with an eigenvalue increases, the modes are characterized by higher spatial wavenumbers, which is characteristic of bluff-body flows. The accuracy of mode reconstruction is similar for Gaussian random projections and single-pixel measurements, as indicated in Fig. 10. However, single pixel measurements may be less expensive and more realistic in real-world applications, and sampling 10% of the original measurements results in a reconstruction accuracy of 90% in this example.

Taking the DMDC results as the reference, Fig. 11 shows the error of the eigenvalues with increasing number of measurements used in cDMDC. Note that the eigenvalues are the same for compressed DMDC and compressed sensing DMDC, as shown in Algorithm 4. The eigenvalue λ_0 corresponding to zero frequency is estimated with the greatest accuracy from the fewest measurements, presumably because this mode contains the most energy in the flow. The error in eigenvalues associated with other modes decreases logarithmically with increasing compression ratio. Single pixel measurements have similar performance compared with Gaussian random measurements for small compression ratios. When $p = n$, the error goes to zero for single pixel measurement, since the measurement is an invertible permutation of the identity matrix, and cDMDC is equivalent to DMDC in this case.

Successful reconstruction using compressed sensing relies on the sparsity of the state in some transform basis. Indeed, both Φ and \mathbf{B} are

sparse in the DCT basis. In general, the DMDC modes are more sparse than the actuation matrix, and we choose $K = 300$ to ensure a good reconstruction of the modes and \mathbf{B} , achieving an error of about 1%. In particular, we use the CoSaMP algorithm [60] to perform the l_1 minimization with 10 iterations and the desired sparsity of $K = 300$. Typically, the sparsity value K is a hyperparameter that must be optimized through cross-validation. However, in compressed sensing, it is often possible to characterize the expected sparsity of representative signals in an offline experiment by computing the Fourier transform of the full signal and evaluating reconstruction accuracy at different levels of truncation. The L^2 errors between the compressive DMDC and DMDC modes are shown in Fig. 10. The convergence of error versus compression ratio in compressed sensing DMDC is much slower than that in compressed DMDC, especially for the zero frequency mode.

The necessary number of measurements is given from theory to be $p \sim 4K \log_{10}(n/K) \approx 2728$, which corresponds to a compression ratio of 5%. This matches the observation that the error curves plateau at around $p/n = 0.05$. Overall, the error for compressed measurements using Gaussian random projections is comparable to using single-pixel measurements.

V. Conclusions

In summary, a unified framework has been presented for compressive system identification based on the dynamic mode decomposition. First, the previously developed compressed sensing dynamic mode decomposition (DMD) and DMD with control algorithms is described in a common mathematical framework,

providing algorithmic implementation details. Next, it is shown how it is possible to construct reduced-order models from compressed measurements and then reconstruct full-state modes corresponding to the reduced states via compressed sensing. This lifting procedure adds interpretability to otherwise black-box models because the reduced states are now associated with full-state modes. The compressed DMD with control algorithm is demonstrated on two example systems, including a high-dimensional discretized simulation of fluid flow past a pitching airfoil. In both cases, accurate modal decompositions are achieved with surprisingly few measurements, showcasing the efficacy of the proposed method. In addition, the entire code base has been released to promote reproducible research and reduce the barrier to implement these methods.

There are a number of important extensions and future directions that arise out of this work. First, it will be interesting to further investigate the relationship between the controllable and observable subspaces and full-state recovery via compressed sensing. The goal is a generalized theory that combines the notion of controllability and observability, based on the structure of the A , B , and C matrices, and the notion of sparse signal recovery, via the structure of the low-rank embedding P . It may also be possible to extend results to nonlinear estimation [72,73] and control [74–76] through the connection to the Koopman operator. In addition, the methods described here are directly applicable to experimental measurements [77] because they do not require access to a model of the system. It may be possible to significantly reduce the required spatial resolution in experiments, improving the effective bandwidth and enabling the characterization of faster flow phenomena. A sparsifying proper orthogonal decomposition basis may be obtained first with non-time-resolved measurements at full resolution. It is then possible to collect many fewer spatial measurements at much higher temporal resolution, identify a reduced-order model, and characterize the full-state modes in the offline library. This also suggests that it may be possible to combine space and time compressed sensing strategies for DMD.

The growing intersection of dynamical systems, machine learning, and advanced optimization are driving tremendous innovations in the characterization and control of complex systems [5,78]. Although data are becoming increasingly abundant, there remain applications, such as feedback flow control, where real-time measurements are costly and control decisions must be made with low latency to ensure robust performance. In these applications, techniques that strategically select sensor data into the most relevant information will enable higher performance in more sophisticated flow control applications. It is likely that the methods developed here may be combined with principled sensor selection methods to promote enhanced sparsity based on learned structures and patterns [55,79–81]. Moreover, it may also be possible to enforce known physics or symmetries in the regression procedure, as in [82], to improve model performance and accelerate learning from data.

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